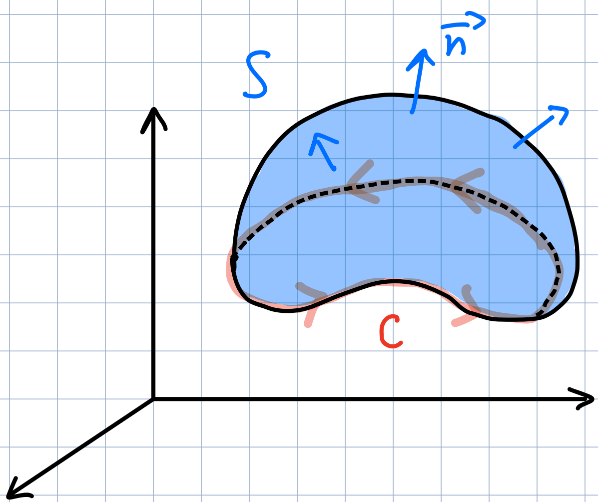
LAST TIME:

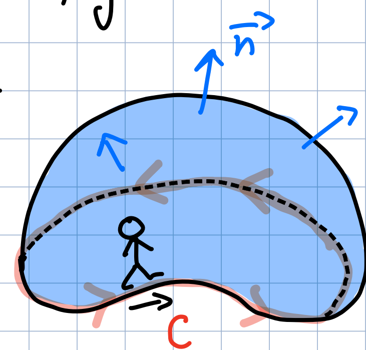
Stoke's theorem



S - oriented surface (choice of $\vec{n}$), bounded by a closed curve C.

C has an "induced orientation":

- walking along C with your head in the direction of $\vec{n}$, you should see S on your left.



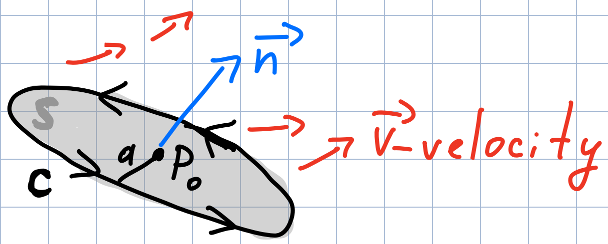
Stoke's THM:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{dS}$$

- line integral of (tangental component) of $\vec{F}$ along the boundary equals the flux of curl through the surface.

Notation: $C = \partial S$

- **Interpretation of curl $\vec{v}$** ← velocity field of a fluid



tiny disc about P_0 of radius a.

Stoke's thm

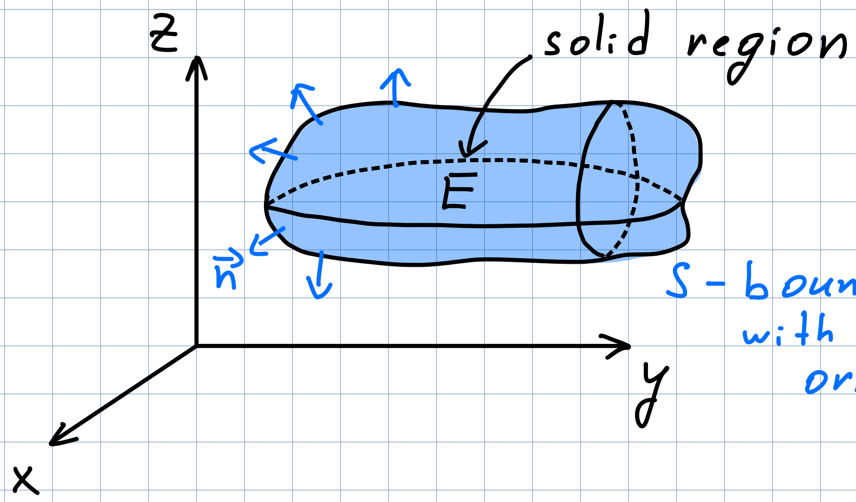
$$\underbrace{\iint_S \operatorname{curl}\vec{v}\cdot\vec{n}\,dS}_{\approx\ \pi a^2 \operatorname{curl}\vec{v}(P_0)\cdot\vec{n}} = \underbrace{\oint_C \vec{v}\cdot d\vec{r}}_{\text{"circulation" of } \vec{v} \text{ around } C}$$

"circulation" of $\vec{v}$ around C – tendency of fluid to move around C

$$\Rightarrow \operatorname{curl}\vec{v}(P_0)\cdot\vec{n} = \lim_{a\to 0} \frac{1}{\pi a^2} \oint_{C_a} \vec{v}\cdot d\vec{r}$$

> The curl of a vector field measures the tendency of that field to rotate around a given point.

Divergence theorem



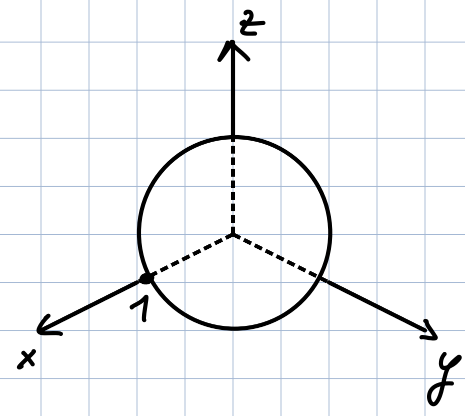
S - boundary surface
with outward (positive) orientation

Divergence thm:

$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{flux of } \vec{F} \text{ through } S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

<u>Ex:</u> S - unit sphere: $x^2+y^2+z^2=1$, $\vec{F}(x,y,z)=\langle z, y, x\rangle$.

Find the flux $\iint_S \vec{F}\cdot d\vec{S}$.



<u>Sol:</u> $\operatorname{div}\vec{F} = \frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z} = 0+1+0 = 1$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} \overset{\text{Divergence}}{\underset{\text{thm}}{=\!=\!=}} \iiint_B \operatorname{div}\vec{F}\, dV$$

unit ball: $x^2+y^2+z^2 \le 1$

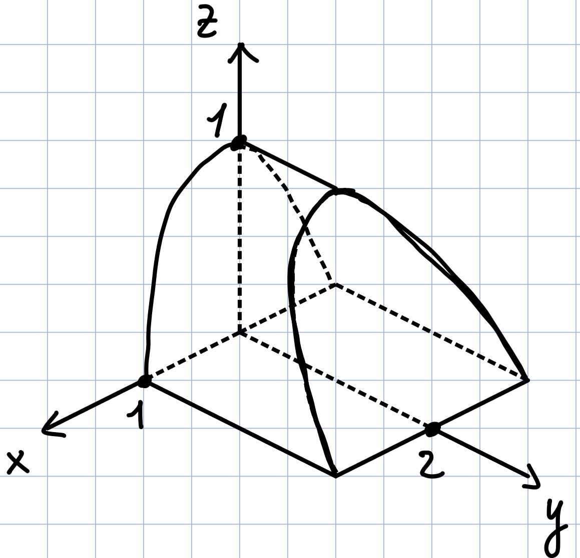
$$= \iiint_B 1\, dV = \text{Volume of } B$$

$$= \frac{4}{3}\pi \cdot 1^3 = \frac{4\pi}{3}$$

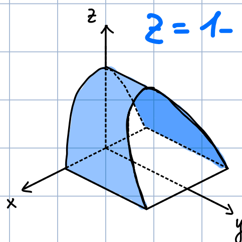
Ex: S - boundary surface of E bounded by: $z = 1 - x^2$ - parabolic cylinder

and $z = 0,\ y = 0,\ y + z = 2$ - planes

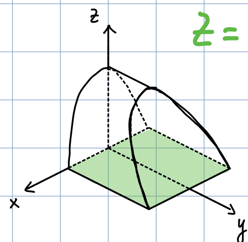
$\vec{F} = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \rangle$. Find $\iint\limits_S \vec{F} \cdot d\vec{S}$.



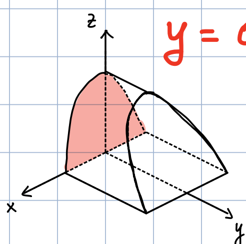
$z = 1 - x^2$



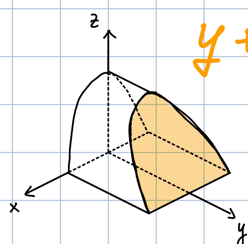
$z = 0$



$y = 0$



$y + z = 2$



Sol: 1) $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}\sin(xy) = y + 2y$

2) E - type 3 solid region: $\begin{cases} -1 \le x \le 1 \\ 0 \le z \le 1 - x^2 \\ 0 \le y \le 2 - z \end{cases}$

3) $$\iint\limits_S \vec{F} \cdot d\vec{S} = \iiint\limits_E 3y\, dV$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \underbrace{\int_0^{2-z} 3y\, dy}_{\frac{3}{2} y^2 \Big|_{y=0}^{y=2-z}} dz\, dx$$

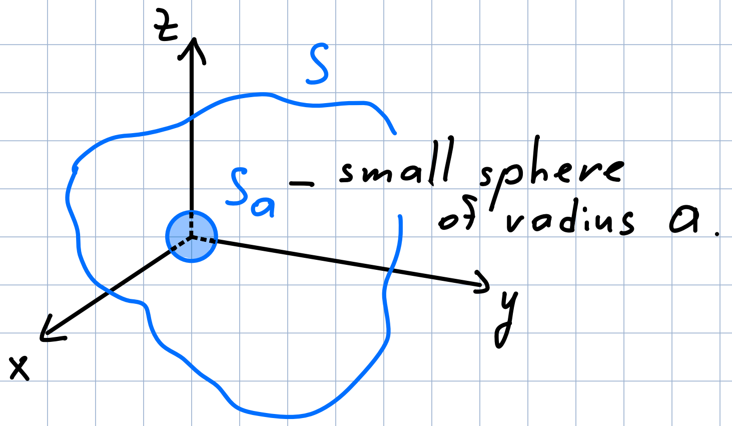
$$= \int_{-1}^{1} \underbrace{\int_0^{1-x^2} \frac{3}{2}(2-z)^2\, dz}_{-\frac{1}{2}(2-z)^3 \Big|_{z=0}^{z=1-x^2} = -\frac{1}{2}\left((1+x^2)^3 - 8\right)} dx = -\frac{1}{2}\int_{-1}^{1}(x^6 + 3x^4 + 3x^2 - 7)\, dx$$

$$= \frac{184}{35}$$

Ex: $\vec{E}(\vec{r}) = \frac{\varepsilon Q}{|\vec{r}|^3}\vec{r}$ - electric field with a charge Q at the origin.

$\vec{r} = \langle x, y, z \rangle$

Show that the flux through any surface S enclosing the origin is $\iint_S \vec{E}\cdot d\vec{S} = 4\pi\varepsilon Q$.



Sol: $$\iint_S \vec{E}\cdot d\vec{S} = \underbrace{\iint_{S_a} \vec{E}\cdot d\vec{S}}_{\frac{\varepsilon Q}{a^3}\underbrace{a}_{\vec{r}\cdot\vec{n}}\cdot 4\pi a^2 = 4\pi\varepsilon Q} + \underbrace{\iiint_V \operatorname{div}\vec{E}\, dV}_{=0}$$

$$\operatorname{div}\vec{E} = \varepsilon Q \operatorname{div}\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

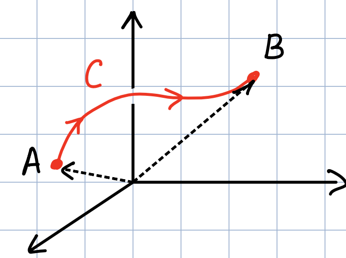
$$\left.\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) &= (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}\,2x^2(x^2+y^2+z^2)^{-5/2} \\
\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) &= (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}\,2y^2(x^2+y^2+z^2)^{-5/2} \\
\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) &= (x^2+y^2+z^2)^{-3/2} - \frac{3}{2}\,2z^2(x^2+y^2+z^2)^{-5/2}
\end{aligned}\right\} \Longrightarrow$$

$$\operatorname{div}\vec{E} = \varepsilon Q\left(3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2}\right) = 0$$

Summary of Stoke's-like theorems:

1. Fund. thm for line integrals:

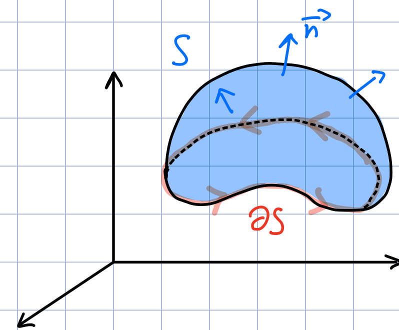
$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$



2. Stokes thm:

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

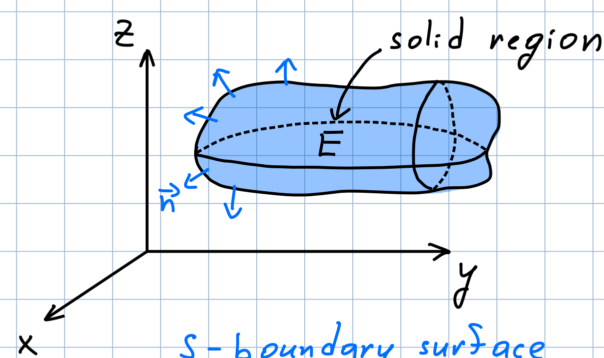
↑boundary curve



3. Divergence thm:

$$\iiint_E \operatorname{div} \vec{F} \, dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}$$

↑boundary surface



S - boundary surface with outward (positive) orientation